Announcements

1) Quiz Tuesday, practice problems on Canvas.

Recall: We were solving

$$y'' + 3y' + 2y = \sin(t)$$

$$y(0) = 0, \quad y'(0) = 1.$$

We got

$$\mathcal{L}(y)(s) =$$

$$= \frac{-1}{s+2} + \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{5}\frac{1}{s+2} + \frac{1}{10}\frac{1-3s}{s^2+1}$$

$$= \frac{3}{2}\frac{1}{s+1} - \frac{6}{5}\frac{1}{s+2} + \frac{1}{10}\frac{1}{s^2+1} - \frac{3}{10}\frac{s}{s^2+1}$$

Using linearity of the inverse Laplace transform,

$$y(t) = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{6}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{1}{10}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{3}{10}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= \boxed{\frac{3}{2}e^{-t} - \frac{6}{5}e^{-2t} + \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t)}$$

Using a table

Impulse

(Section 7.9)

Idea: models short term, violent interactions (i.e. particle collision, bashing objects with hammers, etc.)

Wishful Thinking

If only there was a function $\delta(t)$ with

$$\int_0^\infty f(t)\,\delta(t)\,dt = f(0)$$

for all continuous functions f. Collision takes place at $t=0$.

If the collision occurs at time T,

$$\int_0^\infty f(t)\,\delta(t-T)\,dt$$

$$= f(T)$$

Dirac's Motivation

Orthonormal bases in finite-dimensional vector spaces

Suppose $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

For example, in $\mathbb{R}^3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For a general vector

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ in } \mathbb{R}^n, \text{ can}$$

write

$$v = \sum_{i=1}^{n} v_i e_i .$$

So for example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot e_1 + 2e_2 + 3e_3$$

In order to capture the $i^{\underline{th}}$ coordinate of v,
dot-product v with e_i:

$$v_i = v \cdot e_i$$

number

vectors

Quantum Mechanics

States - typically functions

with $$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$$

are the objects of quantum mechanics. A State is a measurement of the system.

Measurements = the values of ψ at any given point t.

Hence, we want a function $\delta(t)$ with

$$\boxed{\begin{gathered}\int_{-\infty}^{\infty} \psi(t)\,\delta(t)\,dt = \psi(0), \\ \int_{-\infty}^{\infty} \psi(t)\,\delta(t-T) = \psi(T)\end{gathered}}$$

If ψ, φ are complex-valued functions of a real number t, define

$$\boxed{\psi \cdot \varphi = \int_{-\infty}^{\infty} \psi(t)\,\overline{\varphi(t)}\,dt}$$

- if $\int_{-\infty}^{\infty} |\psi(t)|^2 dt$, $\int_{-\infty}^{\infty} |\varphi(t)|^2 dt$

are finite, the dot product always exists.

We want a function δ

with

$$\boxed{f \cdot \delta_T = f(T)}$$

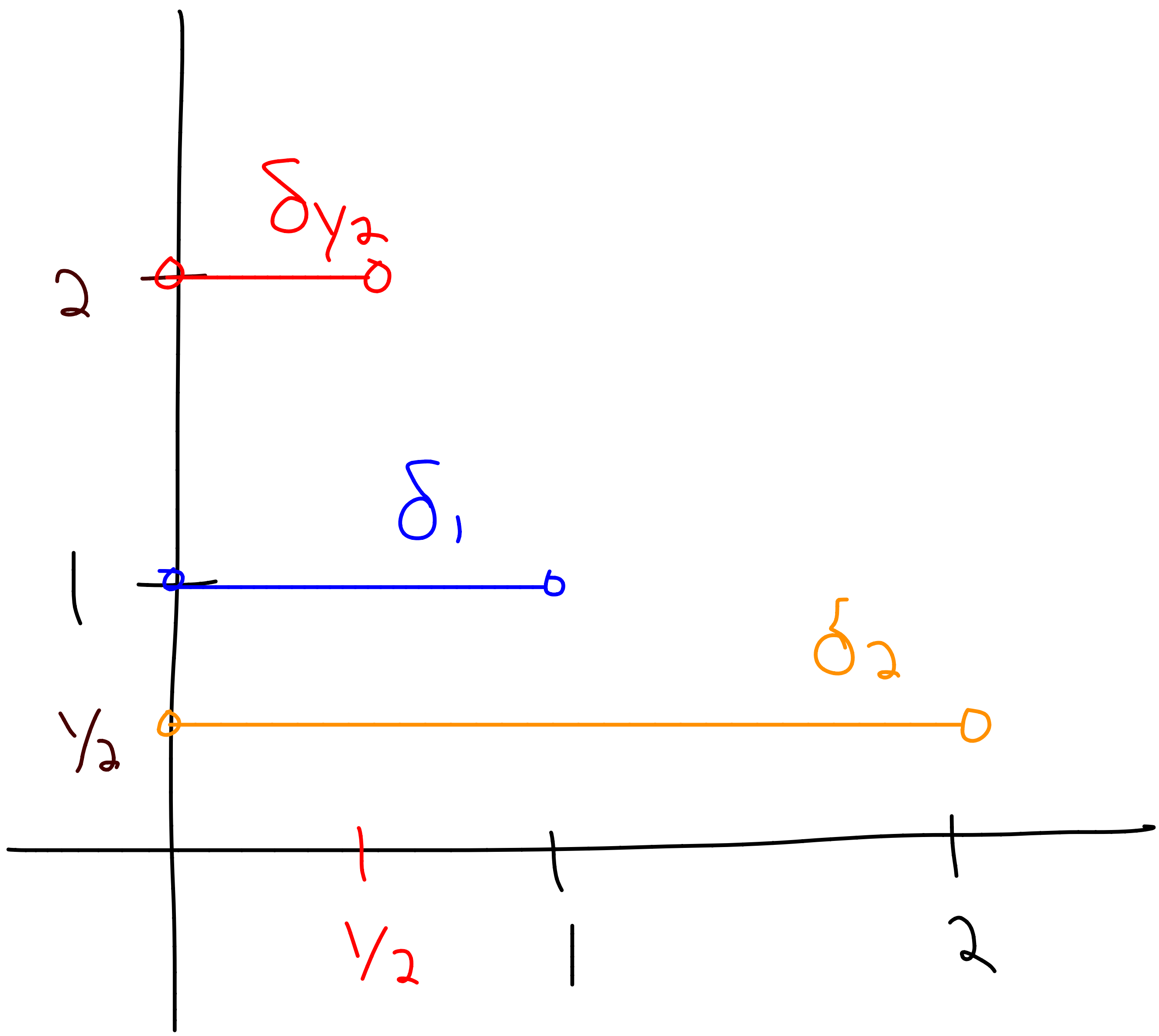
for all admissable functions f.

Preliminaries

Let $\delta_h(x) = \begin{cases} \frac{1}{h}, & 0 < x < h \\ 0, & \text{otherwise} \end{cases}$

h is a real number.

Picture



If f is continuous on $(-\infty,\infty)$,

consider

$$\lim_{\boxed{h \to 0}} \int_{-\infty}^{\infty} f(t)\,\delta_h(t)\,dt$$

$$= \lim_{h \to 0} \int_0^h f(t)\frac{1}{h}\,dt$$

$$= \lim_{h \to 0} \frac{\int_0^h f(t)\,dt}{h}$$

Let g be an antiderivative for f.

By the fundamental theorem of calculus, the limit becomes

$$\lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= g'(0)$$ by definition of the derivative

$$= \boxed{f(0)}$$

Since $g'(t) = f(t)$.

$\delta_h \to \delta$ but not in any way you are familiar with - and δ is not a function!

If δ were a function, it would be zero everywhere but at $t=0$, so the integral against δ would be zero.

δ is a measure or a distribution.

Nevertheless (Laplace transform)

We can write (formally)

$$\mathcal{L}(\delta)(s) = \int_0^\infty e^{-st}\delta(t)\,dt$$

$$= 1.$$

Similarly,

$$\mathcal{L}(\delta_T)(s) = \int_0^\infty e^{-st}\delta(t-T)\,dt$$

$$= e^{-sT}$$

Example 1: A 10kg mass is attached to a spring. The mass is released from rest 2 m below equilibrium. There are no external damping forces on the system and the spring constant is $k = 9$. After $\frac{\pi}{4}$ seconds, the mass is struck by a hammer with a force of 2 N. Find a formula for the position $x(t)$ of the mass.

$$10x''(t) + 9x(t) = 2\delta(t - \pi/4)$$

$$x(0) = 2, \quad x'(0) = 0.$$

Take Laplace Transform of both sides:

$$10\mathcal{L}(x'')(s) + 9\mathcal{L}(x)(s)$$

$$= 2\mathcal{L}(\delta_{\pi/4})(s)$$

$$= 2e^{-\frac{\pi s}{4}}$$

Using properties of the Laplace transform,

$$2e^{-\frac{\pi s}{4}} = 10\left(s^2 \mathcal{L}(x)(s) - s\,\overbrace{x(0)}^{=2} - \underbrace{x'(0)}_{=0}\right) + 9\mathcal{L}(x)(s)$$

$$= 10s^2 \mathcal{L}(x)(s) - 20s + 9\mathcal{L}(x)(s)$$

So solving,

$$2e^{-\frac{\pi s}{4}} + 20s = (10s^2+9)\,\mathcal{L}(x)(s),$$

and

$$\mathcal{L}(x)(s) = \frac{2}{10}\,\frac{e^{-\pi s/4}}{s^2+9/10} + 2\,\frac{s}{s^2+\frac{9}{10}}$$

. . .

$$\mathcal{L}(x)(s) = \frac{2}{10}\,\frac{e^{-\pi s/4}}{s^2+9/10} + 2\,\frac{s}{s^2+\frac{9}{10}}$$

$$= \frac{1}{5}\,\frac{e^{-\pi s/4}}{s^2+\frac{9}{10}} + 2\,\frac{s}{s^2+\frac{9}{10}}$$

Taking the inverse Laplace transform,

$$x(t) = \frac{1}{5}\,\mathcal{L}^{-1}\left(\frac{e^{-\pi s/4}}{s^2+\frac{9}{10}}\right) + 2\cos\left(\frac{3}{\sqrt{10}}t\right)$$

(formula, p. 386)

$$= \boxed{2\cos\left(\frac{3}{\sqrt{10}}t\right) + \frac{1}{5}\sin\left(\frac{3}{\sqrt{10}}\left(t-\pi/4\right)\right)u(t-\pi/4)}$$